MHD Waves in Compressible Magnetically Twisted Flux Tubes

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The oscillatory modes of a **magnetically twisted compressible flux tube embedded in a compressible magnetic environment** are investigated in cylindrical geometry. The **general dispersion equation** in terms of Kummer's functions is obtained for the approximation of weak and uniform internal twist. **The sausage, kink and fluting modes** are examined by means of the derived exact dispersion equation. The solutions of this dispersion equation are found analytically for short and long wavelength limits under plasma conditions representative of the solar photosphere and corona. Numerical solutions for the phase velocity of the allowed eigenmodes are obtained for a wide range of wavenumbers and varying magnetic twist. Our results generalize previous classical and widely applied studies of MHD wave oscillations in magnetic loops with no twist. Applications to solar magneto-seismology are discussed.

Derivation of General Dispersion Equation

The plasma motion is governed by the system of single-fluid, linearised, ideal-MHD equations for a compressible magnetized plasma (see, *e.g.* [1]):

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} + \nabla p + \frac{1}{\mu_0}(\mathbf{b} \times (\nabla \times \mathbf{B_0}) + \mathbf{B_0} \times (\nabla \times \mathbf{b})) = 0, \quad (1)$$

$$p + \boldsymbol{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \boldsymbol{\xi} = 0, \quad (2)$$

$$\mathbf{b} + \nabla \times (\mathbf{B_0} \times \boldsymbol{\xi}) = 0, \quad (3)$$

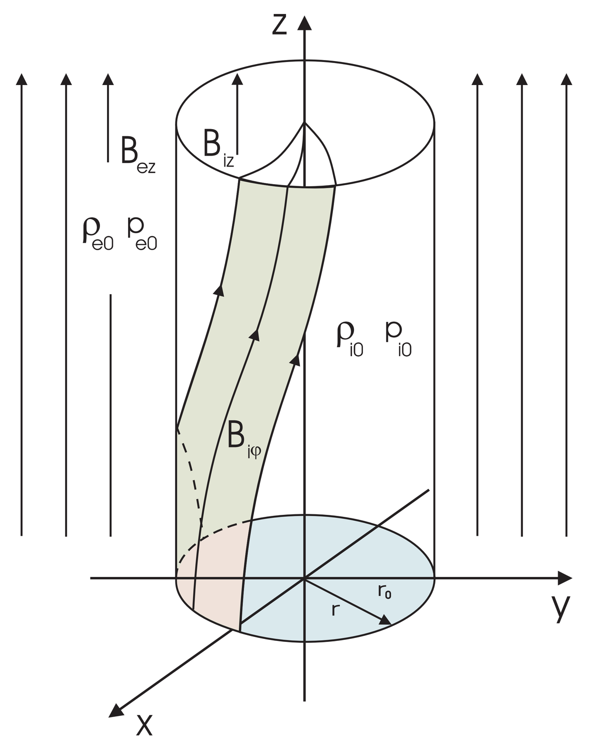

Fig. 1: The geometry of the problem. The straight vertical magnetic flux tube is twisted uniformly. At the boundary of the tube there is a jump in the magnetic twist.

where, in a cylindrical coordinate system with the equilibrium state inhomogeneous in the radial direction only, $\boldsymbol{\xi} = (\xi_r, \xi_\varphi, \xi_z)$ is the Lagrangian displacement vector denoting perturbations from the equilibrium position, $\rho_0(r)$ and $p_0(r)$ are the unperturbed pressure and density, $\mathbf{B_0} = (0, B_{0\varphi}(r), B_{0z}(r))$ is the equilibrium magnetic field, γ is the adiabatic gas index, μ_0 is the magnetic permeability, p is the perturbation of plasma pressure and $\mathbf{b}$ is the perturbation of the magnetic field. The unperturbed state and the geometry of the implemented model is shown in Fig.1. The radius of the tube r_0, and all dependent variables inside the tube have index i, while quantities outside the tube are denoted with index e.

Let us introduce the Eulerian perturbation of total pressure: $p_T = p + \mathbf{B_0}\mathbf{b}/\mu_0$. For the normal-mode analysis in cylindrical geometry the perturbation of all quantities are Fourier decomposed, *e.g.*

$$\xi, p_T \sim \exp[i(kz + m\varphi - \omega t)].$$

Here ω is the mode frequency, m is the azimuthal order of the mode, k is the longitudinal (axial) wavenumber. In a cylindrical equilibrium, the magnetic field and plasma pressure satisfy the equilibrium condition in the radial direction:

$$\frac{d}{dr}\left(p_{i0} + \frac{B_{i\varphi}^2 + B_{iz}^2}{2\mu_0}\right) + \frac{B_{i\varphi}^2}{\mu_0 r} = 0. \quad (4)$$

Here, the second term in the brackets represents magnetic pressure and the third term derives from magnetic tension due to the azimuthal component of the equilibrium magnetic field ($B_{i\varphi}$). We assume that the kinetic pressure (p_{i0}) is constant (though not equal to zero) inside the flux tube. Gravity is not included in the present analysis. The particular case of a uniformly twisted equilibrium magnetic field, ($\mathbf{B_0(r)}$), of the form:

$$\mathbf{B_0(r)} = \begin{cases} (0, Ar, B_{iz}), & r \le r_0, \\ (0, 0, B_{ez}), & r > r_0, \end{cases} \quad (5)$$

is considered, where

$$B_{iz} = B\left(1 - 2\frac{A^2 r^2}{B^2}\right)^{1/2} \quad (6)$$

in order to satisfy the pressure equilibrium governed by Equation (4). Here A and, B are arbitrary constants. This choice of magnetic field may represent solar atmospheric flux tubes with observed weakly twisted field components (see [2]).

Solution inside the flux tube

After some algebra the set of Equations (1)-(3) may be reduced to one equation for the internal Eulerian perturbation of total pressure p_{iT} with variable x:

$$4x^2 \frac{d^2 p_{iT}}{dx^2} + 4x\frac{dp_{iT}}{dx} - \left(m^2 - \left(\frac{n^2-k^2}{knE^{1/2}} + 2(1+m\alpha)\right)x + x^2\right)p_{iT} = 0$$

where

$$x = \frac{n}{k}E^{1/2}k_\alpha^2 r^2 \quad (7)$$

is the normalized variable,

$$n^2 = \frac{\omega^4}{(V_{iA}^2 + C_{iS}^2)(\omega^2 - \omega_{ic}^2)}, \quad E = \frac{4A^4 n^2}{\mu_0^2 D_i^2 k^2 (1-\alpha^2)^2},$$

$$\alpha^2 = \frac{4A^2\omega_{iA}^2}{\mu_0 \rho_{i0}(\omega^2 - \omega_{iA}^2)^2}.$$

$k_\alpha = k(1-\alpha^2)^{1/2}$ is the effective longitudinal wavenumber. Let

$$p_{iT} \sim x^\lambda f(x), \quad (8)$$

where λ is an arbitrary constant and $f(x)$ is an unknown function. If we choose $\lambda = -1/2$, the equation for total pressure takes the form

$$\frac{d^2 f}{dx^2} + \left[-\frac{1}{4} + \frac{\kappa}{x} + \frac{\frac{1}{4} - \mu^2}{x^2}\right] f = 0, \quad (9)$$

where

$$\kappa = \frac{n^2 - k^2}{4knE^{1/2}} + \frac{1}{2}(1 + m\alpha), \quad \mu = \frac{m}{2}. \quad (10)$$

The solutions are known as the linear combination of Whittaker functions:

$$f(x) = C_1^* M_{\kappa,\mu}(x) + C_3^* W_{\kappa,\mu}(x), \quad (11)$$

where C_1^* and C_3^* are arbitrary constants. The solutions can also be written in the form of Kummer functions [3]:

$$M_{\kappa,\mu}(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}+\mu} M\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, x\right), \quad (12)$$

$$W_{\kappa,\mu}(x) = e^{-\frac{1}{2}x} x^{\frac{1}{2}+\mu} U\left(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, x\right). \quad (13)$$

Using the functions (12), (13) and solutions (8), (11), we obtain

$$p_{iT} = C_1^* e^{-\frac{x}{2}} x^{\frac{m}{2}} M(a, b, x) + C_3^* e^{-\frac{x}{2}} x^{\frac{m}{2}} U(a, b, x), \quad (14)$$

where

$$a = \frac{m}{2}(1 - \alpha) - \frac{n^2 - k^2}{4knE^{1/2}}, \quad b = 1 + m. \quad (15)$$

Note, in the center of the flux tube ($r = 0$) the solution of Equation (9) must be finite, therefore, the constant C_3^* is set equal to zero. After some algebra, we arrive at the solution for the internal Lagrangian displacement $\xi_{ir}(x)$:

$$\xi_{ir}(x) = C_1^* \frac{1}{D_i(1-\alpha^2)} e^{-\frac{x}{2}} x^{\frac{m-1}{2}} \left(\frac{nk_\alpha^2 E^{1/2}}{k}\right)^{1/2} \cdot \left[m(1-\alpha)M(a,b,x) + 2xM'(a,b,x)\right], \quad (16)$$

Solution outside the flux tube

Outside the tube, in $r > r_0$, where the magnetic twist is equal to zero and the z component of the unperturbed magnetic field is uniform we can use the solution obtained by [4]:

$$p_{eT} = C_2^* I_m(m_{e0}r) + C_4^* K_m(m_{e0}r), \quad (17)$$

$$\xi_{er} = C_2^* I'_m(m_{e0}r) + C_4^* K'_m(m_{e0}r), \quad (18)$$

where C_2^* and C_4^* are arbitrary constants and I_m, K_m are modified Bessel functions of the first and second kind of order m and

$$m_{e0}^2 = \frac{(k^2 C_{eS}^2 - \omega^2)(k^2 V_{eA}^2 - \omega^2)}{(V_{eA}^2 + C_{eS}^2)(k^2 C_{eT}^2 - \omega^2)}.$$

Because, for the external environment, p_{eT} and ξ_{er} must be finite at infinity ($r \to \infty$), the constant C_2^* is set equal to zero. The dash denotes the derivative of a Bessel function: $K'_m(m_{e0}r) = dK_m/dz$ evaluated at $z = m_{e0}r$.

The dispersion equation and solutions

Applying boundary conditions to the inside and outside solutions, yields the required general dispersion relation:

$$D_e \frac{r_0}{m_{0e}} \frac{K_m(m_{0e}r_0)}{K'_m(m_{0e}r_0)} = -\frac{A^2 r_0^2}{\mu_0^2} + D_i r_0^2 \frac{(1-\alpha^2)}{m(1-\alpha) + 2x_0 M'(a,b,x_0)/M(a,b,x_0)}$$

The dash denotes the derivative of a Kummer function evaluated at $x = x_0$.

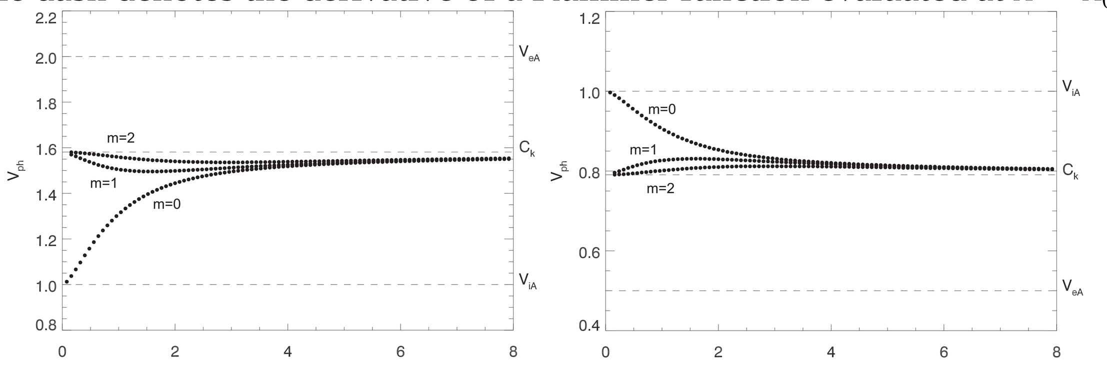

Fig. 2: The normalized phase-speed (V_{ph}) as function of the dimensionless wavenumber kr_0 for cases $V_{Ae} > V_{Ai}$ (left panel) and $V_{Ae} < V_{Ai}$ (right panel) in an incompressible medium. We have taken $\rho_{i0} = \rho_{e0}$ and depict the solution to the dispersion relation for the sausage (m=0), kink (m=1) and fluting (m>1) modes.

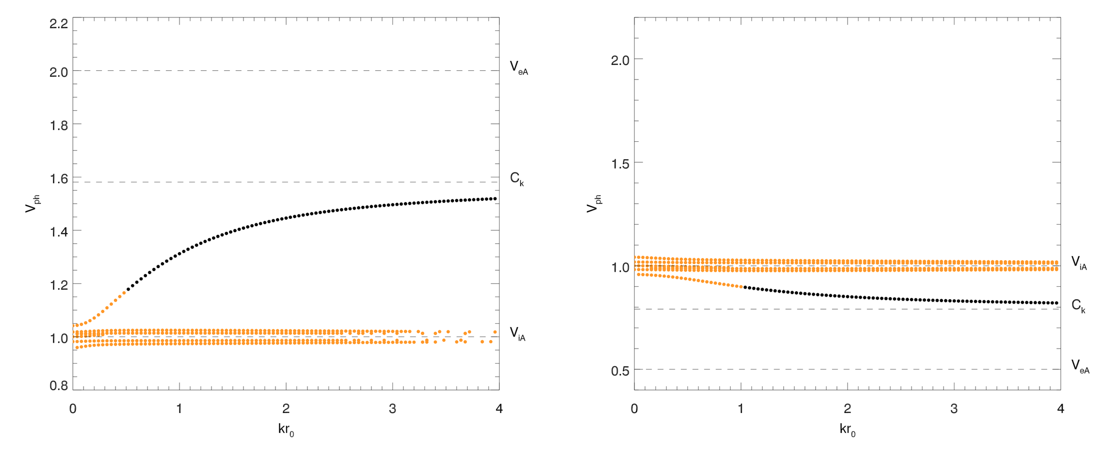

Fig. 3: The normalized phase-speed (V_{ph}) of the m=0 modes in a twisted tube ($V_{Ai\varphi} = 0.1$) as function of kr_0 for cases $V_{Ae} > V_{Ai}$ and $V_{Ai} > V_{Ae}$ are shown at the left and right panels. The surface and body modes are shown. Note the change of character, from body to surface wave, at $kr_0 = 0.5$ (left panel) and $kr_0 = 1$ (right panel).

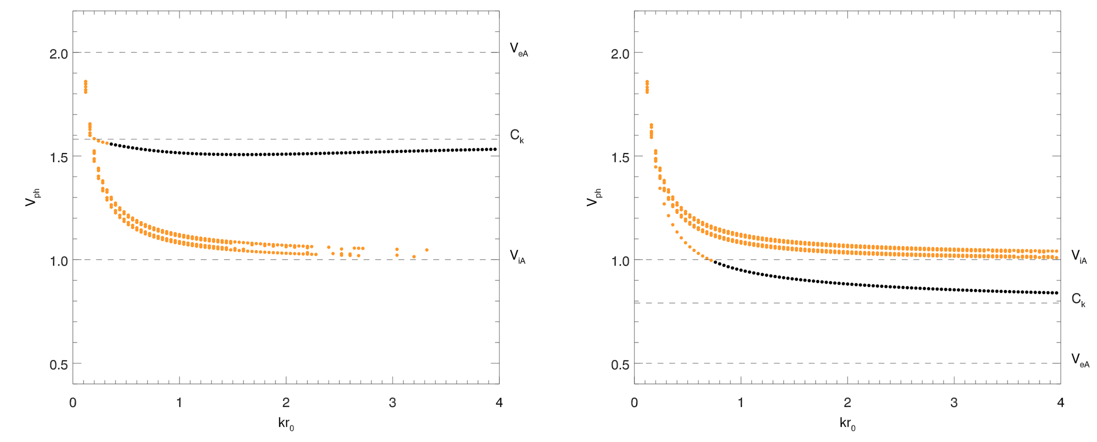

Fig. 4: The normalized phase-speed (V_{ph}) of the m=1 modes in a twisted tube ($V_{Ai\varphi} = 0.1$) as function of (kr_0) for cases $V_{Ae} > V_{Ai}$) and $V_{Ai} > V_{Ae}$. The surface and body modes are shown. Note the band of modes around V_{Ai}, and kink mode which have $V_{ph} \to \infty$ as $kr_0 \to 0$.

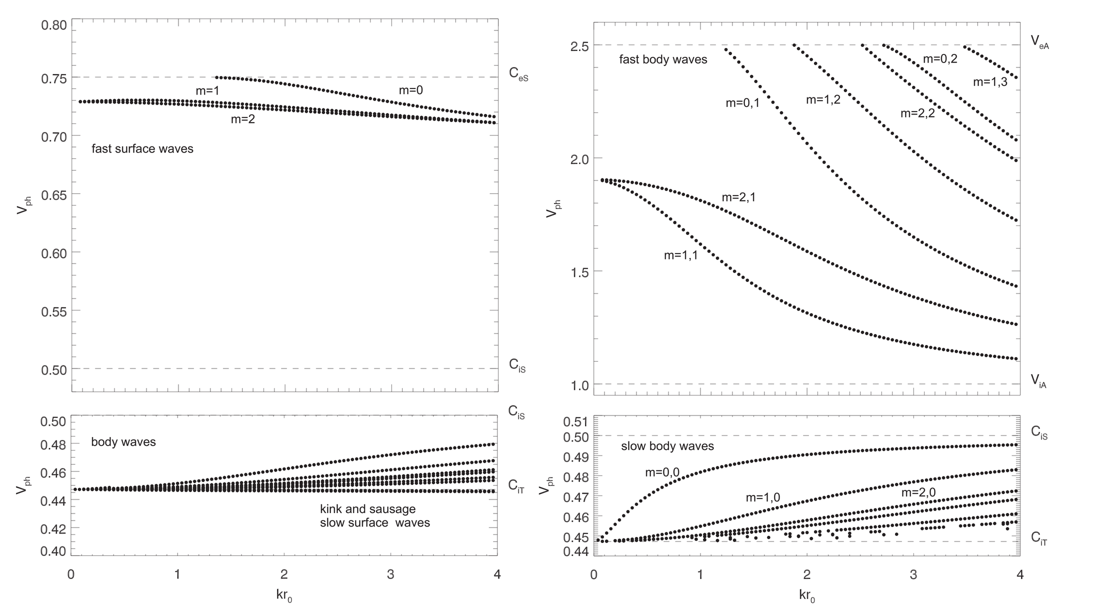

Fig. 5: The normalized phase-speed (V_{ph}) as function of (kr_0) under photospheric (i.e $V_{Ai} > C_{Se} > C_{Si} > V_{Ae}$) and under coronal conditions (i.e $V_{Ae} > V_{Ai} > C_{Si} > C_{Se}$) are shown at the left and right panels of the image, respectively. Here $C_{Se} = 0.75V_{Ai}$, $V_{Ae} = 0.25V_{Ai}$, $C_{Si} = 0.5V_{Ai}$ for photosphere region and $C_{Se} = 0.25V_{Ai}$, $V_{Ae} = 2.5V_{Ai}$, $C_{Si} = 0.5V_{Ai}$ for solar corona. Sausage (m=0), kink (m=1) and fluting (m>1) modes are shown. $m = i, j$: the i refers to the mode (sausage, kink, etc.) and j refers to the branch of the zeroes of eigenfunctions.

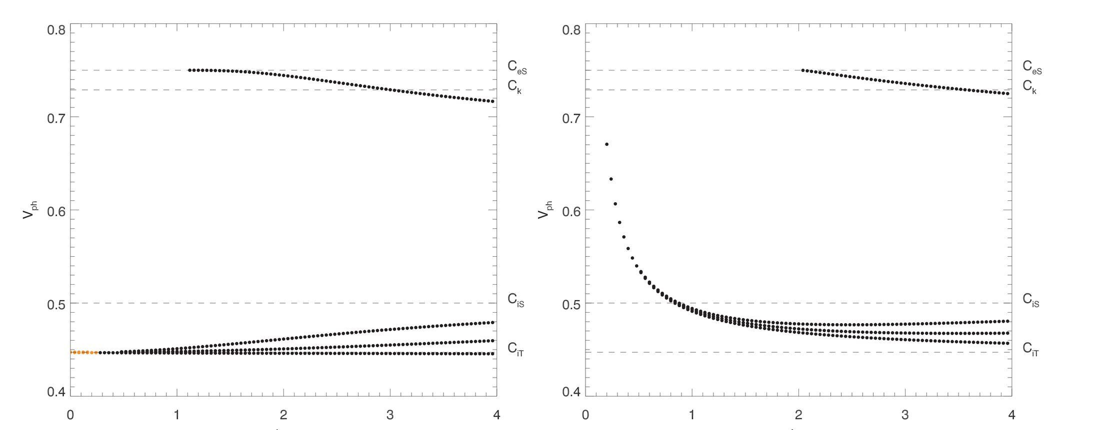

Fig. 6: The normalized phase-speed (V_{ph}) of the sausage mode (m=0) and the kink mode (m=1) as function of (kr_0) for the photosphere (i.e $V_{Ai} > C_{Se} > C_{Si} > V_{Ae}$) are shown at the left and right panels, respectively. The sound and Alfvén speeds are the same as in Fig. 5.

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