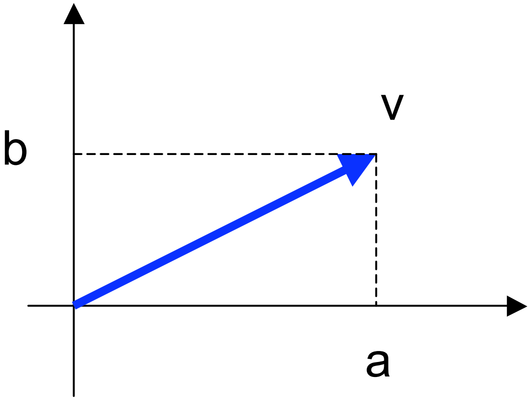
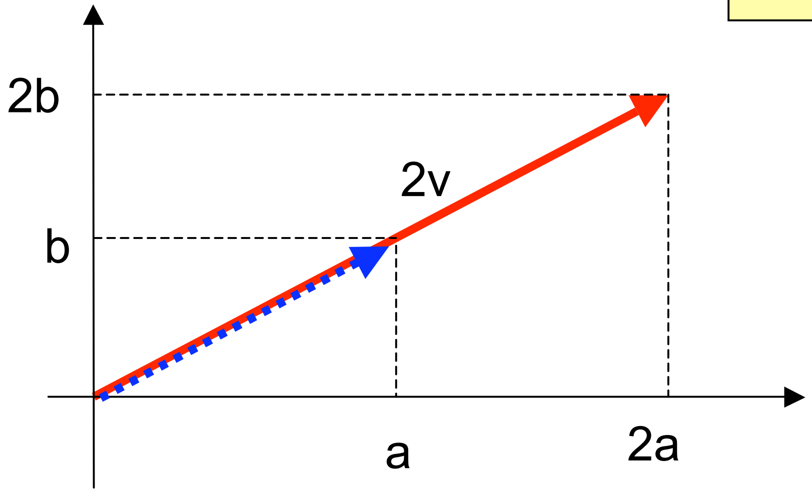
Vectors



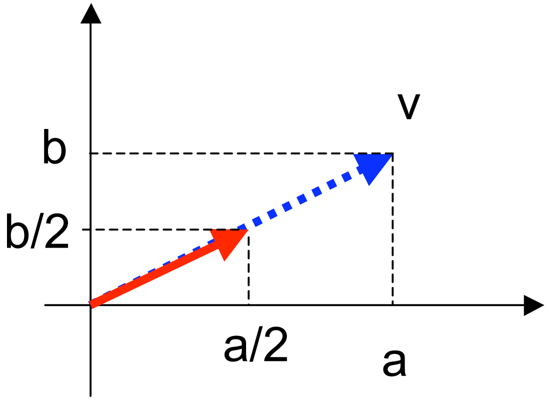
$$v = \begin{bmatrix} a \\ b \end{bmatrix}$$

Multiplied by a constant:

$$kv = k\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ka \\ kb \end{bmatrix}$$



k=2



k=?

Transpose

If $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ then $v^T = \begin{bmatrix} a & b & c \end{bmatrix}$ and vice versa.

Transpose

$$v = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$ What is v^T?

$$v = \begin{bmatrix} 2 & 9 & 0 & -1 \end{bmatrix}$$ What is v^T?

Inner product

(1) of a column vector with itself

$$v^T v = \begin{bmatrix} a & b & c \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \underbrace{a^2 + b^2 + c^2}_{\text{Scalar}}$$

(2) of two column vectors of the same length

$$v^T w = \begin{bmatrix} a & b & c \end{bmatrix} . \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \overbrace{a\alpha + b\beta + c\gamma}$$

If $v = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 5 \end{bmatrix},$ **what is the inner product of**

(i) ***v*** **with itself?**

(ii) ***w*** **with itself?**

(iii) ***v*** **and** ***w*****?**

Matrices

Four columns

Two rows

$$\begin{array}{cc} & \begin{array}{cccc}\downarrow & \downarrow & \downarrow & \downarrow\end{array} \\ \begin{array}{c}\longrightarrow \\ \longrightarrow\end{array} & \begin{bmatrix} 1 & -3 & 5 & 7 \\ 0 & 6 & 4 & -2 \end{bmatrix} \end{array}$$

Dimension: (2x4)

$$A = \begin{bmatrix} 10 & -2 & 4 & 6 & 0 \\ 5 & 4 & -9 & -1 & 3 \\ 7 & 0 & -3 & 8 & -4 \end{bmatrix}$$

What is the dimension of the matrix A?

$$A^T =$$

What is the dimension of the matrix A^T?

Multiplied by a constant

$$kA = k\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \\ ke & kf \end{bmatrix}$$

$$3 * \begin{bmatrix} -2 & 0 \\ 3 & -5 \end{bmatrix} =$$

A square matrix

is a matrix with (number of rows) = (number of columns)

A diagonal matrix

is a square matrix with zero off diagonal terms.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

An identity matrix

is a diagonal matrix with unity diagonal terms.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

dimension?

Square? Diagonal? Identity?

$$\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Matrix and Vector algebra

Addition/subtraction

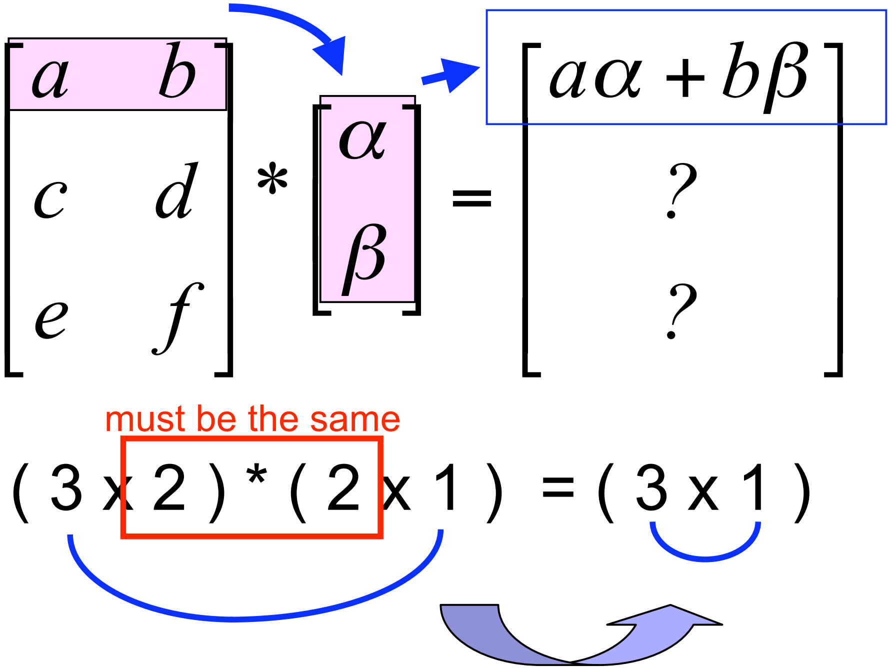
Two matrices (or vectors) can only add or subtract if their dimensions are the same.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 5 & 6 \\ -1 & -2 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & -4 \\ -5 & 9 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = ?$$

Multiplication

Matrices and vectors can only multiply each other if their dimensions are compatible.



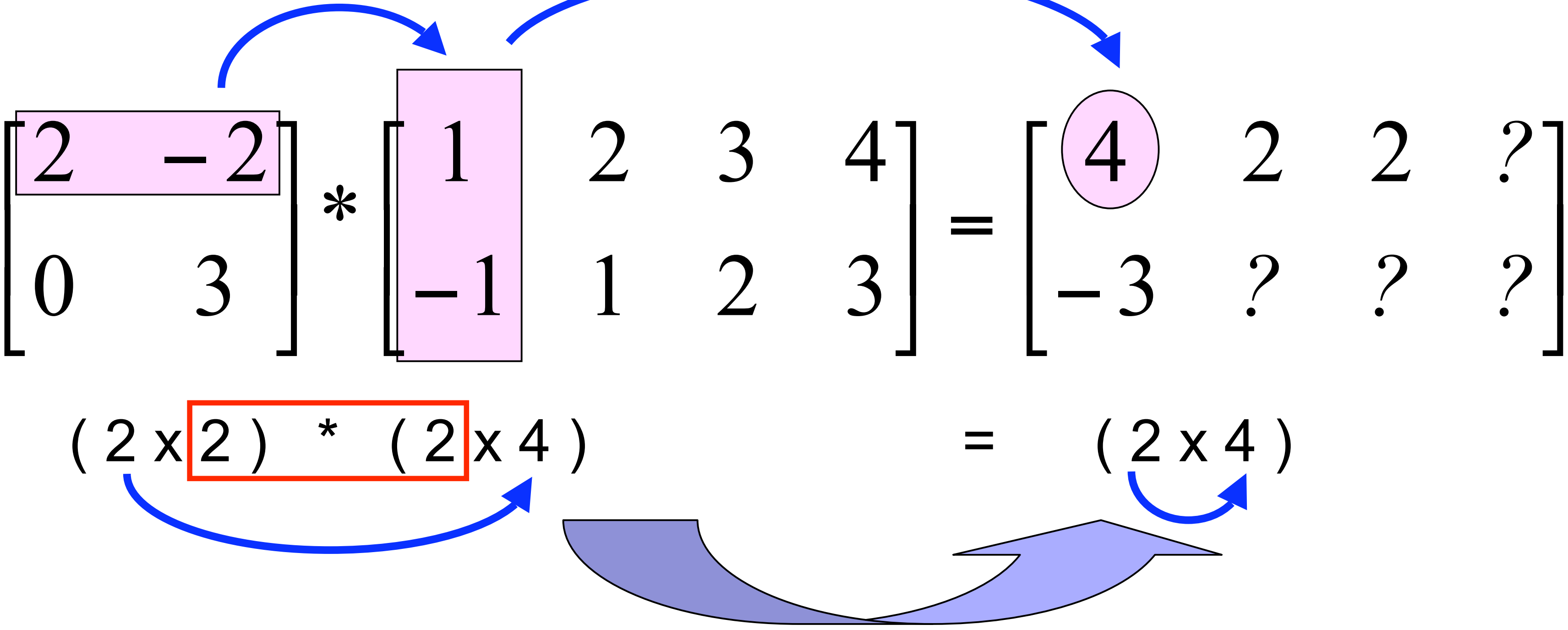
$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} * \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a\alpha + b\beta \\ ? \\ ? \end{bmatrix}$$

must be the same

Dimension: (3 x 2) * (2 x 1) = (3 x 1)

Which of these multiplications can be carried out?

$$\begin{bmatrix} 4 & -2 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} 2 \\ 8 \\ -7 \end{bmatrix} \qquad \begin{bmatrix} -3 \\ 1 \end{bmatrix} * \begin{bmatrix} 2 & 0 \\ -2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \end{bmatrix} * \begin{bmatrix} 9 \\ 4 \\ 6 \end{bmatrix}$$



$$\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 & ? \\ -3 & ? & ? & ? \end{bmatrix}$$

$$(2 \times \boxed{2)\ *\ (2} \times 4) = (2 \times 4)$$

Which of these multiplications can be carried out?

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ -2 & -3 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

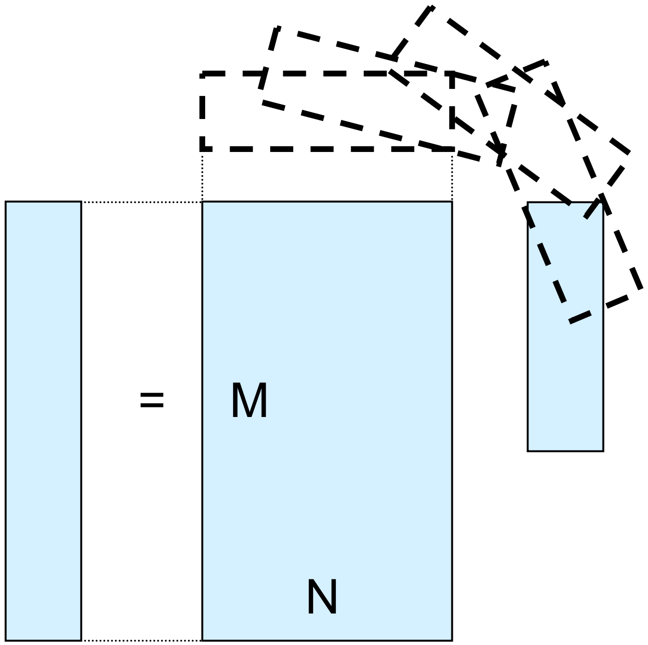
More general forms

Column vector

Column vector

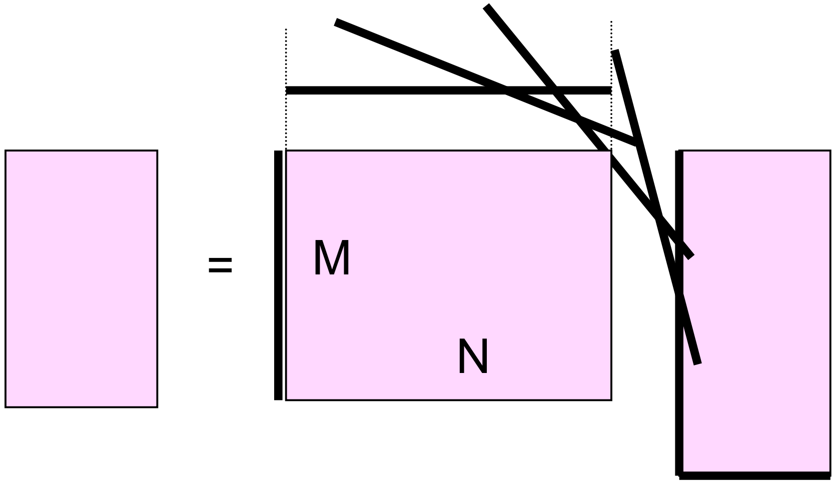
$$y = A x$$

? = (M x N) ?



$$A = B C$$

? = (M x N) (N x P)



Simultaneous equations

$$y_1 = a_1 x_1 + a_2 x_2 + e_1$$
$$y_2 = b_1 x_1 + b_2 x_2 + e_2$$
$$y_3 = c_1 x_1 + c_2 x_2 + e_3$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 x_1 + a_2 x_2 \\ b_1 x_1 + b_2 x_2 \\ c_1 x_1 + c_2 x_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$y = Ax + e$$

Examples on Simultaneous Equations

$$\begin{aligned} y_1 &= -5x_1 + x_2 + e_1 \\ y_2 &= 2x_1 - x_2 + e_2 \end{aligned} \quad \Longrightarrow \quad y = \begin{bmatrix} -5 & 1 \\ 2 & -1 \end{bmatrix} x + e$$

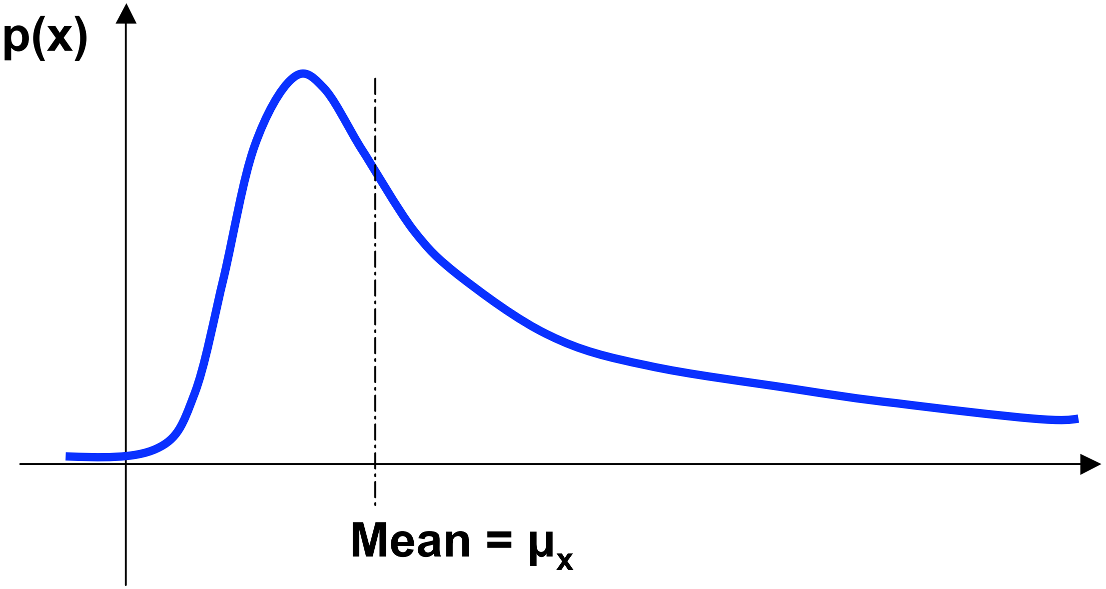
$$\begin{aligned} y_1 &= -4x_1 + x_2 - x_3 + e_1 \\ y_2 &= -2x_2 + e_2 \\ y_3 &= x_1 - 3x_3 + e_3 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} y_1 &= -4x_1 + x_2 - x_3 + e_1 \\ y_2 &= 0x_1 - 2x_2 + 0x_3 + e_2 \\ y_3 &= x_1 + 0x_2 - 3x_3 + e_3 \end{aligned}$$

$$\Downarrow$$

$$y = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} x + e$$

Expected value

For a random variable X with a probability density function



the expected value of X is the mean of the random variable.

$$\boxed{E[X] = \mu_x}$$

Variance

It measures the spread of a random variable X with respect to its mean. It is written using the ‘expected value’ notation:

$$var(X) = E\left[(X - \mu_x)^2\right] = \sigma_x^2$$

Covariance

It measures how much two random variables X and Y vary together. It is written using the ‘expected value’ notation:

$$cov(X,Y) = E\left[(X - \mu_x)(Y - \mu_y)\right] = \sigma_{xy}^2$$

Properties of the ‘expected value’ operation

(1)

$$E[X+Y]=E[X]+E[Y]$$

(2)

$$E[a\,X]=a\,E[X]$$

a constant

a random variable

E.g. If $E[X]=10,$ then $E[5X]=5E[X]=50$

(3) If two random variables X and Y are independent, then

$$E[X\,Y]=E[X]E[Y]$$

Covariance of two *independent* random variables

If two random variables X and Y are independent,
what is the covariance of the two variables? Guess…

By definition: $$cov(X,Y)=E\left[\underbrace{(X-\mu_x)(Y-\mu_y)}\right]$$

Expand: $$(X-\mu_x)(Y-\mu_y)=XY-X\mu_y-\mu_xY+\mu_x\mu_y$$

$$E[XY]=E[X]E[Y]=\mu_x\mu_y$$

$$E[X\mu_y]=E[X]\mu_y=\mu_x\mu_y$$

$$E[\mu_xY]=\mu_xE[Y]=\mu_x\mu_y$$

Therefore: $$cov(X,Y)=E\left[XY-X\mu_y-\mu_xY+\mu_x\mu_y\right]$$

$$=\mu_x\mu_y-\mu_x\mu_y-\mu_x\mu_y+\mu_x\mu_y=0$$

Conclusion: If two random variables X and Y are independent, the covariance between them is zero.

It is often assumed that measurement noise (error) is independent of the measured variable.

Example: X = baby's weight at 6 months

e = associated measurement error, with $E[e] = 0.$

The measurement error is generally independent of the baby's weight.

Hence,

$$\boxed{cov(X, e) = E[(X - \mu_x)(e - 0)] = 0}$$