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Sheffield Economic Research Paper Series

SERPS no. 2025010

ISSN 1749-8368

29 October 2025

Existence of majority equilibria with non-ordered preferences*

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Abstract

Majority voting is widely observed to produce stable policy outcomes, despite theoretical predictions of instability in multidimensional policy spaces. The present paper shows that stability can arise because voters have non-ordered preferences. We model preferences as correspondences within d-dimensional policy spaces and introduce a geometric measure of orderedness based on the angular spread α of strictly preferred alternatives. Our main result is that majority equilibria exist provided $\alpha < \arcsin\sqrt{(d+1)/(2d)}$ with a dimension-free bound at $\alpha < \pi/4 = 45°$. We use Euclidean preferences to show that for random samples of voters with probability one for d tending to infinity there exist majority equilibria provided $\alpha < \arccos\sqrt{1/(m+1)}$. Our findings suggest that modest deviations from fully ordered preferences can ensure stability of collective decisions under the majority rule.

Keywords: ambiguity • centerpoint theorem • collective decision-making • Euclidean preferences • majority equilibrium • non-ordered preferences • status quo bias • voting

JEL-classification: C65, D71, D72, D81

*We are grateful to Georgiy Solomanchuk for carrying out numerical computations, and to audiences at the Universities of East Anglia, Glasgow, Sheffield and York for helpful comments. Crès gratefully acknowledges financial support from the Center for Behavioral Institutional Design and Tamkeen under the NYU Abu Dhabi Research Institute Award CG005.

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1 Introduction

Overview: From a theoretical viewpoint, it is striking that majority voting works so well in practice, electing governments and legislatures that often deliver stable policies. Indeed, when decisions involve many dimensions, the theoretical prediction is that majority voting typically results in instability. To illustrate, consider m self-interested individuals voting on how to divide a cake: no matter what division is proposed, there is always a majority of $m-1$ individuals who would prefer a different one. Hence Tullock's question (Tullock, 1981): Why so much stability?

In the present paper, we suggest that the stability of majority voting in multidimensional settings is reinforced by individual preferences not being fully ordered, that is incomplete, intransitive or both. We take alternatives to have d dimensions and represent an individual preference by a correspondence assigning to each alternative a the set $P(a)$ of strictly preferred alternatives. The individual preference can thus be incomplete, intransitive, subject to ambiguity or reference-point dependence, or characterized by satiation and thick indifference sets. To quantify the degree of "orderedness," we introduce a geometric measure: for all alternatives a, the strictly preferred set $P(a)$ is contained within a cone with vertex at a and angle 2α. When the angle equals $\pi = 180°$, the preference can be rationalized by quasi-concave utility functions; when the angle is zero, the preference can be totally incomplete.

We define a majority equilibrium as an alternative or a status quo for which no majority of voters strictly prefers another alternative. Our main result establishes a relationship between the degree of orderedness and the existence of majority equilibria: if $\alpha < \arcsin\sqrt{(d+1)/(2d)}$ for every voter, then a majority equilibrium exists (Theorem 1). In particular, the condition $\alpha < \pi/4 = 45°$ provides a dimension-free sufficient condition for the existence of such equilibria (Corollary 1). A key insight is that the existence of majority equilibria does not require preferences to be fully non-ordered.

In our model, all voters are assumed to cast their votes: voters who strictly prefer an alternative to the status quo vote for it, while every other voter votes for the status quo. We also introduce the notion of *relative* majority equilibrium, wherein only voters who strictly prefer either the alternative or the status quo cast their votes. All relative majority equilibria are by construction majority equilibria. To get the converse assertion, we extend the description of the preference of a voter by introducing a correspondence $Q(a)$ assigning to each alternative a alternatives to which it is strictly preferred. Within this extended framework, we demonstrate that every majority equilibrium is a relative majority equilibrium (Theorem 2).

Next, we move on to the Euclidean model, where voters are characterized by an ideal point, to illustrate our notion of orderedness. We show how ambiguity about alternatives or ideal points leads to preferences not being fully ordered, and relate the amount of ambiguity to the degree of orderedness. Ambiguity about ideal points can reflect that voters do not know their preferences completely, and ambiguity about alternatives can reflect inattention or that candidates are ambiguous about their platforms.

We also use the Euclidean setup to investigate random electorates, in which ideal points are drawn uniformly on the unit sphere. We focus on two extreme cases: $d \to \infty$ and $d = 2$. For $d \to \infty$, we show that ideal points are nearly orthogonal, which guarantees existence of majority equilibria for almost fully ordered preferences (Theorem 3). For $d = 2$, we provide estimates of the measure of orderedness ensuring existence of a majority equilibrium, and show how fast it converges to fully-ordered preferences when the number of voters increases.

We end the paper with a connection between our main result and the mathematical literature on centerpoints. Following the formalism of Erickson et al. (2009), we show how our findings contribute to improving known conditions for the existence of high-depth points (Theorem 4). This link provides a deeper geometric interpretation of majority equilibria in the Euclidean setting.

History of the problem and related literature: Condorcet's (1785) paradox of voting and Arrow's (1951) impossibility theorem have spurred extensive research on political stability in general and on the existence of a majority equilibrium in particular. Following Plott (1967), the literature explored conditions on individual preferences that guarantee the existence of such an equilibrium. These conditions are highly restrictive, excluding the vast majority of preferences.

For example, Black's (1948) single-peakedness condition allows for 2^n orderings of n alternatives out of the $n!$ possible orderings of n alternatives. Models in which voters have quasi-concave and differentiable utility functions on multi-dimensional policy spaces do not fare any better. As Kramer (1973, p. 285) puts it:

> [...] the various equilibrium conditions for the majority rule are incompatible with even a very modest heterogeneity of tastes, and for most purposes are probably not significantly less restrictive than the extreme condition of complete unanimity of individual preferences.

One route to ensure the existence of a majority equilibrium for general individual preferences is to strengthen the majority rule. Elaborating on the problem of dividing a cake, Greenberg (1979) demonstrates that for general individual preferences and d-dimensional policy spaces, a majority equilibrium exists if and only if the majority required to overturn the status quo exceeds $d/(d+1)$. A limitation of the approach is that if d is large, voters need to almost unanimously strictly prefer an alternative to the status quo to overturn it.

Another route is to impose restrictive distributional assumptions on voters' characteristics: Grandmont (1978) introduces the notion of intermediate preferences and demonstrates that a majority equilibrium exists for symmetric distributions of voters' characteristics. By combining Grandmont's and Greenberg's approaches, Caplin & Nalebuff (1988, 1991) establish the existence of majority equilibria with a super majority rule of 64% for intermediate preferences and a broad class of concave distributions of voters' characteristics.

In broad terms, routes to existence of majority equilibria in a multi-dimensional setup are requiring either quasi-unanimity of individual preferences, quasi-unanimity of the voting rule, strong symmetry conditions, or a mixture of all three. We propose a new route to existence of majority equilibria, namely that individuals hold non-ordered preferences. Our route is supported, at least partially, by experiments showing that individuals have non-ordered preferences, see e.g., Budescu & Weiss (1987), Butlera & Pogrebna (2010), Danan & Ziegelmeyer (2006), Loomes et al. (1989, 1991), Nielsen & Rigotti (2022), Starmer (1999), Tversky (1969), Tversky & Shafir (1992).

It has already been noted that voters' indifference when comparing two alternatives facilitates the aggregation of individual preferences through majority voting. For example, if preferences are dichotomous (every voter partitions alternatives into two classes, where the voter is indifferent between alternatives within each class), then a majority equilibrium exists (Crès, 2001), a result similar to that of approval voting (Brams & Fishburn, 1978). An interesting finding is that the probability of cycles decreases when voters' preferences are allowed to be weak (see, e.g., Fishburn & Gehrlein, 1976; Jones & al., 1995; Van Deemen, 1999; Crès, 2001; Lepelley & Martin, 2001).

Plan of the paper: Section 2 introduces the notion of orderedness and its geometric measure. Section 3 defines the concept of majority equilibrium, presents the main existence result (Theorem 1), and its dimension-free threshold (Corollary 1). Section 4 introduces the notion of a relative majority equilibrium and establishes extended existence results (Theorem 2). Section 5 illustrates our notion of orderedness in an Euclidean framework where alternatives or preferences are ambiguous. Section 6 studies, in the Euclidean setup with random electorates, the relationship between the dimension of the policy space and the degree of orderedness ensuring political stability (Theorem 3). Section 7 connects our findings to the mathematical literature on centerpoints and demonstrates how our results refine and strengthen recent advances in that area (Theorem 4). Finally, in the appendix we present a result on the relationship between the dimension of the decision and the number of voters and the degree of orderedness ensuring political stability (Theorem 5).

2 Setup

Let $A \subset \mathbb{R}^d$ be a convex, compact and non-empty set of alternatives with d being the number of policy issues. There is a finite number of voters $i \in M = \{1,\ldots,m\}$ described by their strict preference correspondences $P_i : A \to A$ with open graphs, $\{\,(x,y) \in A \times A \mid y \in P_i(x)\,\}$ is open for every $i \in M$. Intuitively, $y \in P_i(x)$ can be interpreted as y is strictly preferred to x in case x is the status quo. Therefore, it is possible that $y \in P_i(x)$ and $x \in P_i(y)$, so voter i could support alternative y in case x is the status quo and alternative x in case y is the status quo.

For all $p \in \mathbb{R}^d \backslash \{0\}$ and $\alpha \in [0, \pi/2]$, or in degrees α between 0° and 90°, let $K(p,\alpha) \subset \mathbb{R}^d$ be the open cone with vertex at 0 for which $x \in K(p,\alpha)$ if and only if the angle between x and p is less than α, $\angle x0p < \alpha$. Obviously, $K(p,\alpha)$ is the set of vectors $v \in \mathbb{R}^d$ with $v \cdot p > 0$ for $\alpha = \pi/2$ and $K(p,\alpha) = \emptyset$ for $\alpha = 0$.

Here is how we measure orderedness of preferences: we assume strictly preferred sets are subsets of open cones. Formally, there is $\alpha \in [0, \pi/2]$ such that for every i and all x there are $p_i \in \mathbb{R}^d \backslash \{0\}$ such that

$$P_i(x) \; \subset \; \{x\} + K(p_i, \alpha). \tag{1}$$

The parameter α can seen as a measure of the degree of orderedness of preferences: $\alpha = \pi/2$ can be compatible with convex, complete and transitive preferences; and, $\alpha = 0$ is compatible with totally non-ordered preferences. Preferences can be non-ordered for many reasons, for example because of status quo bias in which case $\alpha = \pi/2$ corresponds to no status quo bias and $\alpha = 0$ corresponds to total status quo bias.

In Section 5 there are two illustrations based on Euclidean preferences satisfying Property (1). The illustrations show how ambiguity about alternatives or preferences can lead to non-ordered preferences.

3 Majority equilibrium

A majority equilibrium is an alternative for which there is no other alternative strictly preferred by more than half of the voters.

Definition 1 *A* ***majority equilibrium*** *is an alternative $\bar{x} \in A$ such that for all alternatives $x \in A$,*

$$|\{\, i \in M \mid x \in P_i(\bar{x}) \,\}| \;\leq\; \frac{1}{2} m.$$

The following theorem, which relates the angles of the cones containing strictly preferred alternatives and existence of majority equilibria, is the main result of the paper.

Theorem 1 *Suppose there is $\alpha \in [0, \pi/2]$ such that for every $i \in M$ and all $x \in A$, $P_i(x) \subset \{x\} + K(p_i, \alpha)$ for some $p_i \in \mathbb{R}^d \backslash \{0\}$. If $\alpha < \arcsin\sqrt{(d+1)/(2d)}$, then there is a majority equilibrium.*

Proof: Let the correspondence $P : A \to A$ map alternatives to sets of alternatives preferred by more than half of the voters

$$P(x) \;=\; \bigcup_{|C| > \frac{m}{2}} \bigcap_{i \in C} P_i(x).$$

Then x is a majority equilibrium if and only if $P(x) = \emptyset$. Moreover, the graph of P is open because the graph of P_i is open for every i. Therefore, according to the theorem in Gale and Mas-Colell (1975) there is an alternative $\bar{x}$ such that either $P(\bar{x})$ is empty or $\bar{x}$ is in the convex hull of $P(\bar{x})$: i.e. either $P(\bar{x}) = \emptyset$ or $\bar{x} \in \operatorname{co} P(\bar{x})$. We show that $\bar{x} \notin \operatorname{co} P(\bar{x})$, so $P(\bar{x}) = \emptyset$.

Take $y, z \in P(\bar{x})$. There are sets of voters C_y and C_z with $|C_y|, |C_z| > m/2$ such that $y \in \cap_{i \in C_y} P_i(\bar{x})$ and $z \in \cap_{i \in C_z} P_i(\bar{x})$. Since $|C_y|, |C_z| > m/2$, the intersection of C_y and C_z is not empty: $C_y \cap C_z \neq \emptyset$, so there is $i \in C_y \cap C_z$ implying $y, z \in P_i(\bar{x})$. There is $\alpha \in [0, \pi/2]$ such that for every i, $P_i(\bar{x}) \subset \{\bar{x}\} + K(\bar{p}_i, \alpha)$ for some $\bar{p}_i \in \mathbb{R}^d \backslash \{0\}$. Hence, the angle between y and z is less than 2α. Let $\tilde{y}$ and $\tilde{z}$ be the normalized alternatives after A is translated to $A - \{\bar{x}\}$, $\tilde{y} = (1/\|y - \bar{x}\|)(y - \bar{x})$ and $\tilde{z} = (1/\|z - \bar{x}\|)(z - \bar{x})$. Then the distance between $\tilde{y}$ and $\tilde{z}$ is less than $2\sin\alpha$. Therefore, the distance between any pair of normalized points in $P(\bar{x})$ is less than $2\sin\alpha$, so the distance between any pair of normalized points in the closure of $P(\bar{x})$ is less than or equal to $2\sin\alpha$.

Let $\tilde{P}(\bar{x})$ be the normalized alternatives in $P(\bar{x})$. According to Jung's Theorem, see Jung (1901) and Berger (2009), there is a closed ball with radius less than or equal to

$$2\sqrt{\frac{d}{2(d+1)}}\sin\alpha \;=\; \sqrt{\frac{2d}{d+1}}\sin\alpha$$

such that the closure of $\tilde{P}(\bar{x})$ is contained in that closed ball. The radius of the ball is less than one, because $\sin\alpha \;<\; \sqrt{(d+1)/(2d)}$. Hence there is $q \in \mathbb{R}^d \setminus \{0\}$ such that $\tilde{y} \in \tilde{P}(\bar{x})$ implies $q \cdot \tilde{y} < 0$ so $y \in P(\bar{x})$ implies $q \cdot y < q \cdot \bar{x}$. Consequently, $\bar{x}$ is not in the convex hull of $P(\bar{x})$, so $P(\bar{x})$ is empty. □

Remark: The assumption in Theorem 1 states that the angle between any pair of points in $P_i(x)$ must be less than: $\pi = 180^\circ$ for $d = 1$; $2\pi/3 = 120^\circ$ for $d = 2$; and so on with limit $\pi/2 = 90^\circ$ for d tending to infinity. Consequently, if $\alpha_i < \pi/4 = 45^\circ$ for every i, then there is a majority equilibrium independently of the number of policy issues d. *End of remark*

Corollary 1 *Suppose there is $\alpha \geq 0$ such that for every i and all x, $P_i(x) \subset \{x\}+K(p_i,\alpha)$ for some $p_i \in \mathbb{R}^d \backslash \{0\}$. If $\alpha < \pi/4$, then there is a majority equilibrium.*

It is remarkable that we provide a dimension-free bound, just as Caplin & Nalebuff (1988) do.

4 Relative majority equilibrium

Some voters may be indifferent or not able to compare some pairs of alternatives x and y and, consequently, may not participate in the vote between these two alternatives. By excluding these individuals, we arrive at the notion of *relative* majority equilibrium. To define it, we require more information about preferences than what is given by strict preference correspondences alone. Specifically, we need correspondences mapping each alternative to the set of alternatives to which it is strictly preferred.

Suppose voter i is described by two correspondences: the strict preference correspondence $P_i : A \to A$; and, the correspondence $Q_i : A \to A$ mapping x to the set of alternatives to which x is strictly preferred, where $y \in Q_i(x)$ means voter i strictly prefers x to y in case x is the status quo.

Definition 2 *A **relative majority equilibrium** is an alternative $\bar{x}$ such that for all alternatives $x \in A$,*

$$|\{\, i \in M \mid x \in P_i(\bar{x}) \,\}| \;\leq\; \frac{1}{2}|\{\, i \in M \mid x \in P_i(\bar{x}) \cup Q_i(\bar{x}) \,\}|.$$

Obviously, if an alternative is a relative majority equilibrium, then it is a majority equilibrium too. At a (relative) majority equilibrium the number voters strictly preferring an alternative is compared to the number (of a subset) of voters.

Three assumptions ensure that majority equilibria are relative majority equilibria: $x \notin Q_i(z)$ with $x \neq z$ implies $(1-\lambda)x+\lambda z \in P_i(z)$ for all $\lambda \in (0,1)$; $P_i(z) \cap Q_i(z) = \emptyset$; and, P_i and Q_i have open graphs. The first assumption is a convexity assumption: convex combinations of the status quo and an alternative, which is not worse than the status quo, is strictly preferred to the status quo. The second assumption is a consistency assumption: an alternative cannot be both better than and worse than the status quo. The third assumption is a technical assumption.

The three assumptions imply that majority equilibria are relative majority equilibria too.

Theorem 2 *Suppose $\bar{x}$ is a majority equilibrium. Then $\bar{x}$ is a relative majority equilibrium.*

Proof: Since $\bar{x}$ is a majority equilibrium, $|\{\, i \in M \mid x \in P_i(\bar{x}) \,\}| \leq m/2$ for all $x \in A$. Suppose $\bar{x}$ is not a relative majority equilibrium. Then there is $x \in A$ such that

$$|\{\, i \in M \mid x \in P_i(\bar{x}) \,\}| \;>\; \frac{1}{2}|\{\, i \in M \mid x \in P_i(\bar{x}) \cup Q_i(\bar{x}) \,\}|.$$

Clearly, $|\{\, i \in M \mid x \in Q_i(\bar{x}) \,\}| < m/2$. For all $y = (1-\lambda)x + \lambda\bar{x}$ with $\lambda \in (0,1)$,

$$\begin{aligned} |\{\, i \in M \mid y \in P_i(\bar{x}) \,\}| &\geq m - |\{\, i \in M \mid x \in Q_i(\bar{x}) \,\}| \\ &> m - \frac{m}{2} \\ &= \frac{m}{2} \end{aligned}$$

contradicting that $\bar{x}$ is a majority equilibrium. $\square$

5 Euclidean preferences

The Euclidean distance on $\mathbb{R}^d$ between two points x and y is $\|x-y\| = \sqrt{\sum_k (x^k - y^k)^2}$. Voters have Euclidean preferences provided: every voter i has an *ideal point* $x_i \in A$ that anchors their preferences over all alternatives; and, an alternative is strictly preferred to another alternative provided it is closer to the ideal point than the other alternative measured by the Euclidean distance.

Definition 3 (Euclidean preferences) *For alternatives a and b voter i strictly prefers b to a if and only if b is closer to x_i than a:*

$$b \;\in\; P_i(a) \iff \|x_i - b\| \;<\; \|x_i - a\|.$$

For $B(x,r)$ being the closed ball with center x and radius r, the set of strictly preferred alternatives to a is the interior of the ball with center at x_i and radius $\|x_i - a\|$, $P_i(a) = \operatorname{int} B(x_i, \|a - x_i\|)$. The natural application of Property (1) to Euclidean preferences is

$$P_i(a) \;\subset\; \operatorname{int} B(x_i, \|x_i - a\|) \cap (\{x_i\} + K(x_i - a, \alpha_i)). \tag{2}$$

Below we consider two takes on non-ordered Euclidean preferences. They both satisfy Property (2) and are motivated by ambiguity.

Illustration 1: Ambiguous alternatives with Euclidean preferences

As a first take on introducing non-orderedness in the Euclidean model, we allow for *sets of alternatives* instead of alternatives. This can reflect that alternatives are opaque to voters, perhaps because candidates are ambiguous about their platforms or voters are inattentive. For simplicity, the ambiguity sets are closed balls centered at alternative a, and their radii are proportional to the distances between the alternatives and ideal points, with a common ambiguity ratio $\eta \in [0,1]$. For alternative a, voter i and ambiguity ratio $\eta \in [0,1]$, let $A_i(a)$ be the closed ball with center at a and radius $\eta\|x_i - a\|$,

$$A_i(a) \;=\; B(a, \eta\|x_i - a\|).$$

Obviously, $A_i(a)$ is $\{a\}$ for $\eta = 0$, and $A_i(a)$ is the closed ball with center at a and radius $\|x_i - a\|$ for $\eta = 1$.

Definition 4 (Ambiguous alternatives) *For two alternatives a and b, voter i strictly prefers b to a if and only if all points in $A_i(b)$ are closer than all points in $A_i(a)$ to the ideal point x_i:*

$$b \;\in\; P_i(a) \iff \forall a' \in A_i(a), b' \in A_i(b) : \|x_i - b'\| \;<\; \|x_i - a'\|.$$

For $\eta = 0$ preferences are Euclidean and for $\eta = 1$ no pair of alternatives can be compared.

Figure 1 illustrates the case $d = 2$ with the interior of the green ball being the set of alternatives strictly preferred to a. Alternative b is preferred to alternative a if and only if

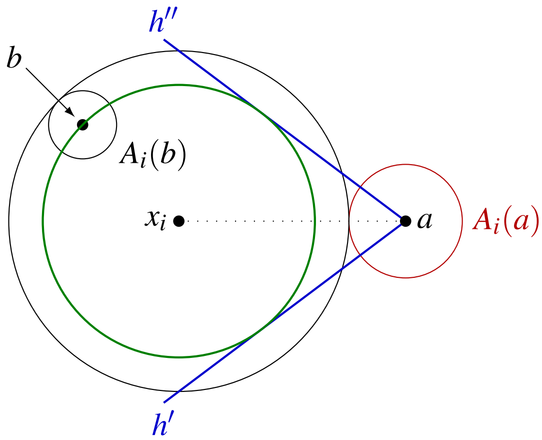

Figure 1: The preferred set with ambiguous alternatives.

the distance from x_i to the farthest point in $A_i(b)$ is smaller than the distance from x_i to the closest point in $A_i(a)$:

$$(1+\eta)\|x_i - b\| < (1-\eta)\|x_i - a\|.$$

Hence,

$$P_i(a) \;=\; \operatorname{int} B\left(x_i, \frac{1-\eta}{1+\eta}\|x_i - a\|\right).$$

So $P_i(a)$ is a subset of a cone with vertex at a containing the half-lines h' and h'':

$$P_i(a) \subset \{a\} + K\left(x_i - a, \arcsin\frac{1-\eta}{1+\eta}\right)$$

making the angle of the cone $\alpha = \arcsin((1-\eta)/(1+\eta))$, which is decreasing in η.

Remark: Another interpretation could be given to the strict preference correspondence represented in Figure 1. For status quo a, there is a cost for every voter i in considering an alternative b; and this cost depends linearly on $\|x_i - b\|$ with common ratio $\gamma \geq 0$.

Definition 5 (Costly alternatives) *For two alternatives a and b:*

$$b \in P_i(a) \iff \|x_i - b\| + \gamma\|x_i - b\| < \|x_i - a\|.$$

For $\gamma = 0$ preferences are Euclidean and for $\gamma = +\infty$ no pair of alternatives can be compared. Obviously:

$$P_i(a) = \operatorname{int} B\left(x_i, \frac{1}{1+\gamma}\|x_i - a\|\right) \subset \{a\} + K\left(x_i - a, \arcsin\frac{1}{1+\gamma}\right).$$

End of remark

Illustration 2: Ambiguous Euclidean preferences

As a second take on introducing non-orderedness in the Euclidean model, we allow voters to have *sets of ideal points* instead of unique ideal points. It is akin to a decision makers having a set of priors instead of a single prior in models of ambiguity à la Bewley (2002). For simplicity, the ambiguity sets are closed balls centered at x_i, and their radii are proportional to the distances between the alternatives and x_i, with common ambiguity ratio $\delta \in [0,1]$: For alternative a, voter i and ambiguity ratio δ, let $B_i(a)$ be the closed ball with center x_i and radius $\delta\|x - a\|$,

$$B_i(a) = B(x_i, \delta\|x_i - a\|).$$

Obviously, $B_i(a)$ is $\{x_i\}$ for $\delta = 0$, and $B_i(a)$ is the closed ball with center at x_i and radius $\|x_i - a\|$ for $\delta = 1$.

Definition 6 (Ambiguous Euclidean preferences) *For two alternatives a and b, voter i strictly prefers b to a if and only if b is closer than a to all ideal points for a:*

$$b \in P_i(a) \iff \forall x \in B_i(a) : \|x - b\| < \|x - a\|.$$

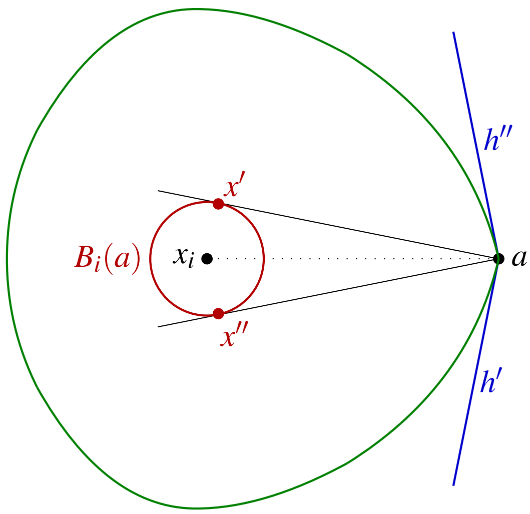

Figure 2: The preferred set with ambiguous preferences.

For $\delta = 0$ preferences are Euclidean and for $\delta = 1$ no pair of alternatives can be compared.

Figure 2 illustrates the case with $d = 2$. First, the (black) cone with vertex at a and half-lines tangent to $B_i(a)$ (at x' and x'') has angle $\arcsin\delta$: it is $K(x_i - a, \arcsin\delta)$. Second, for b to be strictly preferred to a, $b-a$ must be above the hyperplane orthogonal to $2x-a-b$ for all $x \in B_i(a)$:

$$\forall x \in B_i(a) : (2x-a-b)\cdot(b-a) \ > \ 0.$$

Hence the strictly preferred set to a, $P_i(a)$, is the interior of the green set which is a subset of the blue cone with vertex at a containing the half-lines h' (orthogonal to $x'-a$) and h'' (orthogonal to $x''-a$):

$$P_i(a) \ \subset \ \{a\} + K(x_i - a, \arccos\delta). \tag{3}$$

So the angle of the cone $\alpha = \arccos\delta$ is decreasing in δ.

Illustration of the majority equilibrium in the Euclidean model: the Fermat point

For two policy issues $d = 2$ and three voters $m = 3$, there exists a majority equilibrium for $\alpha = \pi/3 = 60^\circ$, which corresponds to $\eta = (2-\sqrt{3})/(2+\sqrt{3}) \approx 0.0718$ in case of ambiguous alternatives and $\delta = 1/2$ in case of ambiguous preferences.

Figure 3 illustrates the case, where voters have ideal points forming a triangle with all angles at vertices smaller than $2\pi/3$. The point F, where

$$\angle x_1 F x_2 \ = \ \angle x_1 F x_3 \ = \ \angle x_2 F x_3 \ = \ \frac{2\pi}{3},$$

is the unique majority equilibrium. F is the *Fermat* point of the triangle which minimizes

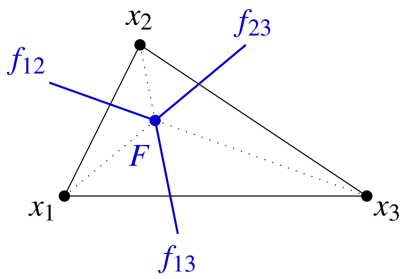

Figure 3: The majority equilibrium with ambiguity.

the sum of the distances to the three ideal points:

$$F \;=\; \arg\min_{f \in \mathbb{R}^2} \|f - x_1\| + \|f - x_2\| + \|f - x_3\|.$$

Let the half-line f_{12}, respectively f_{13} and f_{23}, bisect the angle $\angle x_1 F x_2$, respectively $\angle x_1 F x_3$ and $\angle x_2 F x_3$. The alternatives strictly preferred to F by x_1 are a subset of the cone $\{F\} + K(x_1 - F, 2\pi/3)$ (with vertex at F and delimited by f_{12} and f_{13}). Any pair of the three cones $\{F\} + K(x_i - F, 2\pi/3)$, for $i \in \{1,2,3\}$, having empty intersection, there is no alternative supported by a coalition of two or more voters.

Note that when the angle at a vertex is larger than $2\pi/3 = 120°$, then the corresponding vertex is the Fermat point, and it is a majority equilibrium.

6 The relation between d and α for Euclidean preferences

In the present section we use Euclidean preferences and study the relationship between the number of policy issues d and the degree of orderedness α ensuring existence of majority equilibria in case ideal points are random. We consider two extremes: first the number of policy issues tending to infinity; and, second the number of policy issues being fixed and equal to two. From Theorem 1 follows that for the worst case scenario, the degree of orderedness is decreasing in the number of policy issues and converges to $\pi/4 = 45°$.

The angle α for d tending to infinity

The unit-sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$ is endowed with the uniform distribution. Ideal points of voters are assumed to be a sample of m independent and identically distributed random variables $X_d = \{x_{d,1}, \ldots, x_{d,m}\}$ for every d.

Lemma 1 *For $(X_d)_{d \in \mathbb{N}}$ with probability one,*

$$\lim_{d \to \infty} \max_{i \neq k} |x_{d,i} \cdot x_{d,k}| \;=\; 0.$$

Proof: Follows from Theorem 6 in Cai et al. (2013). □

The lemma implies that the angle between every pair of ideal points converges to 90° as d tends to infinity.

Theorem 3 *For $(X_d)_{d\in\mathbb{N}}$ with probability one for d tending to infinity:*

- *The center of the unit-sphere* 0 *is a majority equilibrium provided*

$$\alpha \;<\; \arccos\sqrt{\frac{2}{m+1}}.$$

- *The average ideal point $c_d = (1/m)\sum_i x_{d,i}$ is a majority equilibrium provided*

$$\alpha \;<\; \arccos\sqrt{\frac{1}{m+1}}.$$

Proof: To ease notation we drop d as subscript. According to Lemma 1 $\lim_{d\to\infty}, \max_{i\neq j}|x_i \cdot x_j| = 0$. Consider $N \subset M$ with $|N| = n$ and $i \in N$. The alternative being supported by voters in N is their average ideal point $(1/n)\sum_{k\in N} x_k$ for both c^d or 0 being the status quo.

For the centre of the unit-sphere 0:

$$\lim_{d\to\infty} x_i \cdot \frac{1}{n}\sum_{k\in N} x_k \;=\; \frac{1}{n}$$

with $\lim_{d\to\infty}\|(1/n)\sum_{k\in N} x_k\|^2 = 1/n$. Consequently,

$$\lim_{d\to\infty} \angle x_i 0 \frac{1}{n}\sum_{k\in N} x_k \;=\; \arccos\frac{1}{\sqrt{n}}.$$

The result follows from setting n equal to $\lfloor 0.5m\rfloor+1$ and using that $\lfloor 0.5m\rfloor+1 \geq 0.5(m+1)$.

For the average ideal point $(1/m)\sum_j x_j$:

$$\lim_{d\to\infty}\left(x_i - \frac{1}{m}\sum_j x_j\right)\cdot\left(\frac{1}{n}\sum_{k\in N} x_k - \frac{1}{m}\sum_j x_j\right) \;=\; \frac{1}{n} - \frac{1}{m}$$

with $\lim_{d\to\infty}\|x_i - (1/m)\sum_j x_j\|^2 = 1-1/m$ and $\lim_{d\to\infty}\|(1/n)\sum_{k\in N} x_k - (1/m)\sum_j x_j\|^2 = 1/n - 1/m$. Consequently,

$$\lim_{d\to\infty} \angle x_i \frac{1}{m}\sum_j x_j \frac{1}{n}\sum_{k\in N} x_k \;=\; \arccos\sqrt{\frac{m-n}{(m-1)n}}.$$

The result follows from setting n equal to $\lfloor 0.5m\rfloor+1$ and using that $m - \lfloor 0.5m\rfloor - 1 \leq 0.5(m-1)$ and $\lfloor 0.5m\rfloor+1 \geq 0.5(m+1)$. □

Remark: Theorem 3 implies that with high probability for d large:

- 0 is a majority equilibrium for: $\alpha < 64.76°$ for $m = 10$; $\alpha < 81.91°$ for $m = 100$; and, $\alpha < 87.44°$ for $m = 1000$.

- c^d is a majority equilibrium provided: $\alpha < 72.45°$ for $m = 10$; $\alpha < 84.29°$ for $m = 100$; and, $\alpha < 88.19°$ for $m = 1000$.

Going from α to m, c^d is a majority equilibrium for $\alpha \leq 75°$ in case $m \geq 14$ and 0 is a majority equilibrium for $\alpha < 75°$ in case $m \geq 29$. The calculations indicate that the deviation from fully complete preferences ensuring political stability is modest provided the number of policy issues is large. *End of remark*

One aspect of Theorem 3 is particularly remarkable. In much of the social choice literature, high-dimensional policy spaces are viewed as an adversary of stability. For example, Greenberg's (1979) supermajority result becomes vacuous as $d \to \infty$, the Caplin & Nalebuff (1988) supermajority threshold rises with d, and, our bound in Theorem 1 falls with dimensionality. There seems to be a 'curse of dimensionality' also in social choice. Theorem 3 offers an opposite view: when the policy space is high dimensional, two randomly chosen vectors are almost surely orthogonal, and this geometric property can foster stability; indeed majority equilibria emerge with only a modest degree of non-orderedness.

Another notable feature of Theorem 3 is that it does not rely on symmetry of ideal points, unlike Grandmont (1978). With probability one, the distribution of ideal points is not symmetric about the center of the unit sphere. In fact, as $d \to \infty$, the inner product of any ideal point with the sum of all ideal points converges to one: $\lim_{d\to\infty} x_i \cdot \sum_k x_k = 1$ for every i. Hence the set of ideal points can be separated from the origin by a hyperplane orthogonal to that sum. Theorem 3 therefore rests on preferences being not fully ordered, rather than on symmetry in the distribution of ideal points.

Lastly, Lemma 1 holds when both the number of policy issues d and the number of voters m tend to infinity, provided $(\log m)/d$ goes to zero. Under these conditions, with probability one the degree of orderedness needed to ensure existence of a majority equilibrium converges to $\pi/2$. We expand on that in the appendix. We study sequences of samples of random ideal points for which both d and m tend to infinity (not necessarily with $(\log m)/d \to 0$). Moreover, majority equilibria are generalized to ρ-majority equilibria according to which an alternative is stable provided there is no other alternative preferred by more than $\rho \times 100\%$ of the voters. We show that with probability one for d and m tending to infinity the centre of the unit-sphere is a ρ-majority equilibrium for all $\alpha < 90°$. Hence, just a pinch of non-orderedness ensures political stability.

The angle α for $d = 2$

Consider samples of m independent and identically distributed random variables $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_m\}$ on the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$. We define the largest distance from the center 0 to the convex hull of the ideal points of voters in C, where C is a subset of $\lfloor m/2 \rfloor + 1$ voters:

$$\tilde{\delta} = \max_{C \subset M, |C| = \lfloor m/2 \rfloor + 1} \min_{x \in \operatorname{co} \tilde{X}_C} \|x\| \tag{4}$$

for $\tilde{X}_C = \{\tilde{x}_i, i \in C\}$. Note that $\tilde{X}_C \subset \{0\} + K(p, \tilde{\alpha})$ for some $p \in \mathbb{R}^2$ and $\tilde{\alpha} = \arccos \tilde{\delta}$.

To estimate $\tilde{\alpha}$ for every m, we generate n independent realizations $\hat{X}_1, \ldots, \hat{X}_n$ of the m-point set $\tilde{X}$. For every realization k, we compute the corresponding $\hat{\alpha}_k = \arccos \hat{\delta}_k$ with $\hat{\delta}_k$ defined according to (4). Then the estimator is:

$$\hat{\alpha} = \frac{1}{n} \sum_{k=1}^{n} \hat{\alpha}_k$$

The graphs in Figure 4 shows the values of the estimator $\hat{\alpha}$ (blue curve) and the 10% and 90% quantiles (yellow curves) as functions of the size of the electorate m for $n = 1000$ realizations. The angle ensuring that the center of the distribution is a majority equilibrium

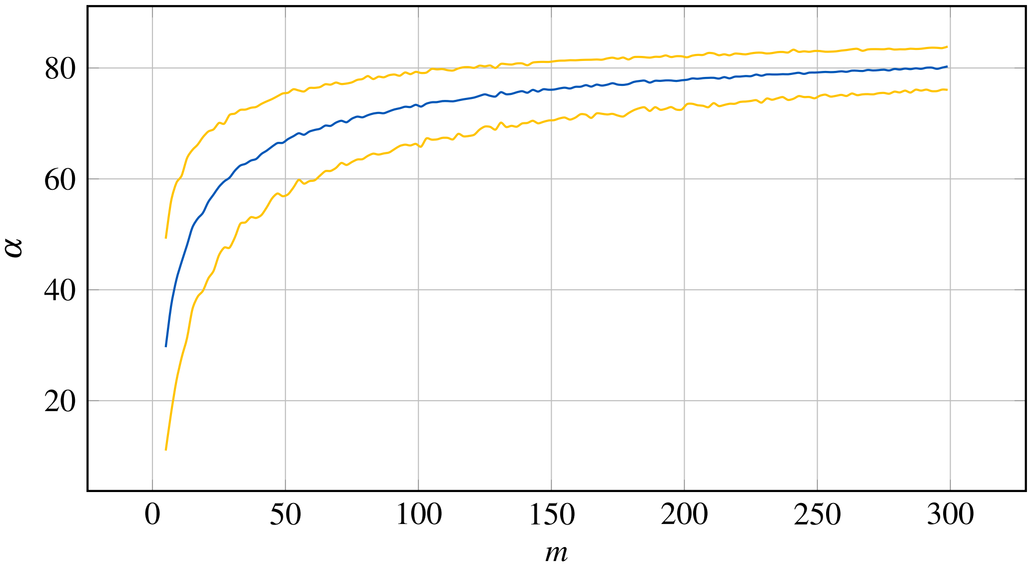

Figure 4: $\hat{\alpha}$ (blue curve) and the 10% and 90% quantiles (yellow curves).

converges quite fast to 90°. For $m = 50$ voters, it is 67°, with the 10% and 90% quantiles at 57° and 75° respectively; and for $m = 300$ voters, it is 80°, with the 10% and 90% quantiles at 76° and 84° respectively.

The angle ensuring that the mean point of each realization $\hat{X}$ is a majority equilibrium is a bit larger. For $m = 50$ voters, it is on average 72°, instead of 67°; and for $m = 300$ voters, it is on average 82° instead of 80°.

7 Centerpoint theorems

Let us now link our framework to the literature on centerpoints, and demonstrate how Theorem 1 strengthens recent advances in that area.

Suppose X is a set in $\mathbb{R}^d$ with m points. Then a point $x \in \mathbb{R}^d$ is a *centerpoint* of X provided every closed half-space containing x contains at least $m/(d+1)$ points of X. There can be multiple centerpoints and they do not need to belong to X. The Centerpoint Theorem states that every finite point set in $\mathbb{R}^d$ has a centerpoint.

The notion of centerpoints has been generalized to cones or wedges in Erickson et al. (2009). Let $\bar{K}(p,\alpha)$ denote the closure of $K(p,\alpha)$. First, define the α-cone *depth* of $x \in \mathbb{R}^d$ with respect to X as

$$D_\alpha(x,X) = \min_{p\in\mathbb{R}^d} |X \cap (\{x\}+\bar{K}(p,\alpha))|.$$

A centerpoint of X is a point x for which $D_{\pi/2}(x,X) \geq \lceil m/(d+1) \rceil$. Second, for fixed α, d and m define the *minmax* depth as

$$\mu_\alpha^d(m) \;=\; \min_{X\in\{N\subset\mathbb{R}^d \,|\, |N|=m\}} \; \max_{x\in\mathbb{R}^d} D_\alpha(x,X).$$

The Centerpoint Theorem states that $\mu_{\pi/2}^d(m) = \lceil m/(d+1) \rceil$. Erickson et al. (2009) show that for $d=2$:

$$\mu_\alpha^2(m) = \lceil m/2 \rceil \text{ for } 2\pi/3 \leq \alpha < \pi,$$

and for an arbitrary d:

$$\mu_\alpha^d(m) = \lceil m/2 \rceil \text{ for } \pi/2+\arccos\sqrt{1/d} \leq \alpha < \pi. \tag{5}$$

Theorem 1 allows us to improve Condition (5), as we get the following corollary.

Theorem 4 *If $\alpha > \pi/2+\arccos\sqrt{(d+1)/(2d)}$, then $\mu_\alpha^d(m) \geq \lceil m/2 \rceil$.*

Clearly, Theorem 4 improves Condition (5), as

$$\arccos\sqrt{\frac{d+1}{2d}} \;<\; \arccos\sqrt{\frac{1}{d}}$$

for $d \geq 2$ with $\lim_{d\to\infty} \arccos\sqrt{(d+1)/(2d)} = \pi/4$ and $\lim_{d\to\infty} \arccos\sqrt{1/d} = \pi/2$.

To prove Theorem 4, we link the notion of centerpoint for wedges to the notion of majority equilibrium in the following Euclidean setup: For every voter i with ideal point $x_i \in \mathbb{R}^d$, the strictly preferred correspondence is defined by

$$P_i(a) \;=\; \{a\} + K(x_i - a, \alpha). \tag{6}$$

Let us first observe that $x_i \in \{a\}+K(p,\alpha)$ if and only if $a+p \in \{a\}+K(x_i-a,\alpha)$: voter i strictly prefers $a+p$ to a.

Consider a society of m voters with ideal points in $X = \{x_1,\ldots,x_m\}$ and common degree of orderedness α. The policy getting the most support against a is in a direction p that maximizes $|X \cap (\{a\}+K(p,\alpha))|$, or equivalently minimizes $|X \cap (\{a\}+\bar{K}(-p,\pi-\alpha))|$. Hence the depth $D_{\pi-\alpha}(a,X)$ is the minimum number of voters that does not support any challenger against a. For a given X, the most stable incumbent is the alternative with maximum depth. Finally, the worst m-points set is one that minimizes this maximum depth, implying that $\mu_\alpha^d(m)$ is a general threshold for existence of a political equilibrium.

We have the following lemma.

Lemma 2 *If $\bar{x}$ is a majority equilibrium, then $D_{\pi-\alpha}(\bar{x},X) \geq m/2$.*

Proof: Suppose to the contrary $D_{\pi-\alpha}(\bar{x},X) < m/2$; then there exists $p \in \mathbb{R}^d$ such that $|X \cap (\{\bar{x}\}+\bar{K}(p,\pi-\alpha))| < m/2$. Hence $|X \cap (\{\bar{x}\}+K(-p,\alpha))| > m/2$. Let $C \subset M$ denote the coalition of voters with ideal points in this set. Then $(\bar{x}-p) \in \cap_C P_i(\bar{x})$, which is a contradiction. □

Lemma 2 implies Theorem 4.

Proof of Theorem 4: Consider an m-point set X, and interpret it as the ideal points of an electorate preferences defined by 6, with a commmon degree of orderedness $\alpha' < \arcsin\sqrt{(d+1)/2d}$. From Theorem 1 we know that there is a majority equilibrium $\bar{x}$ in the convex hull of X. Then by Lemma 2, we get: $D_{\pi-\alpha'}(\bar{x},X) \geq \lceil m/2 \rceil$. Hence $\max_{x\in\mathbb{R}^d} D_\alpha(x,X) \geq \lceil m/2 \rceil$ (with $\alpha = \pi - \alpha'$).

This holds for all m-point sets X. As a consequence $\mu_\alpha^d(m) \geq \lceil m/2 \rceil$ when $\alpha > \pi - \arcsin\sqrt{(d+1)/2d} = \pi/2 + \arccos\sqrt{(d+1)/(2d)}$. □

8 Final remarks

We have proposed a novel possible explanation of why majority voting works so well in practice, namely non-orderedness of preferences. Whereas the classic literature has relied on restrictive symmetry conditions, homogeneity of preferences, or strengthened majority rules, we show that even modest deviations from fully ordered preferences can ensure the existence of majority equilibria.

We model non-orderedness by requiring that, for any incumbent, the set of strictly preferred alternatives lies within a cone of angle $\alpha < 90°$, and we establish sharp angular thresholds guaranteeing the existence of majority equilibria – most notably a dimension-free bound at $\alpha < 45^\circ$. In doing so, we provide a precise, analytically tractable framework explaining why stability can emerge without symmetry assumptions or supermajority rules.

Our analysis leads us to conjecture that the dimension-free stability threshold of $\alpha < 45^\circ$ established in Corollary 1 might be improved to $\alpha < 60^\circ$ with the worst-case scenario arising for three voters in a bidimensional Euclidean setup as illustrated in Figure 3. Moreover, the figure suggests that, in the Euclidean framework, the Fermat point of voters' ideal points might be a better proxy for the point that maximizes political stability than both the mean ideal point and the centre of the distribution of ideal points are.

References

Arrow, K.J.: Social choice and individual values, Wiley (1951).

Berger, M.: Geometry I, Springer Verlag (2009).

Bewley, T.: Knightian decision theory: Part I. Decisions in Economics and Finance **25** (2002), pp. 79–110.

Black, D.: On the rationale of group decision-making, Journal of Political Economy **56** (1948), pp. 23–34.

Brams, S.J.; Fishburn, P.C.: Approval Voting, The American Political Science Review, **72(3)** (1978), pp. 831–847.

Budescu, D.; Weiss, W.: Reflection of transitive and intransitive preferences: A test of prospect theory, Organizational Behavior and Human Decision Processes **39** (1987), pp 184–202.

Butler, D. J.; Pogrebna, G.: How many things can you be indifferent to? The use of a ranking procedure in the measurement of preferences, Theory and Decision **69** (2010), pp 609–628.

Cai, T.; Fan, J.; Jiang, T.: Distributions of Angles in Random Packing on Spheres, Journal of Machine Learning Research **14** (2013), pp. 1837–1864.

Caplin, A.; Nalebuff, B.: On the 64%-majority rule, Econometrica **56** (1988), pp. 787–814.

Caplin, A.; Nalebuff, B.: Aggregation and social choice - a mean voter theorem, Econometrica **59** (1991), pp. 1–23.

Condorcet, M.J.: Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix (1785), Paris.

Crès, H.: Aggregation of Coarse Preferences, Social Choice and Welfare **18** (2001), pp. 507–525.

Danan, E.; Ziegelmeyer, A.: Are preferences complete? An experimental measurement of indecisiveness under risk, Theory and Decision **61** (2006), pp 345–373.

Erickson, J.; Hurtado, F.; Morin, P.: Centerpoint theorems for wedges, Discrete Mathematics and Theoretical Computer Science **11** (2009), pp. 45–54.

Fishburn, P.C.; Gehrlein, W.V.; Fishburn: The paradox of voting: Effects of individual indifference and intransitivity, Journal of Public Economics **14** (1980), pp. 83–94.

Gale, D.; Mas-Colell, M.: An equilibrium existence theorem, Journal of Mathematical Economics **2** (1975), p. 9–15.

Gehrlein, W.V.; Fishburn, P.C.: The probability of the paradox of voting: A Computable Solution, Journal of Economic Theory **13** (1976), pp. 1–25.

Grandmont, J.-M.: Intermediate preferences and majority-rule, Econometrica, **46** (1978), pp. 317–330.

Greenberg, J.: Consistent majority rules over compact sets of alternatives, Econometrica **47** (1979), pp. 627–636.

Jones, B.; Radcliff, B; Taber, C.; Timpone R.: Paradox of voting: Probability calculations for weak preference orders, American Political Science Review **89** (1995), pp. 137–144.

Jung, H.: Ueber die kleinste Kugel, die eine räumliche Figur einschliesst, Journal für die reine and angewandte Matematik **123** (1901), pp. 241–57.

Kramer, G.H.: On a class of equilibrium conditions for the majority rule, Econometrica **41** (1973), pp. 285–297.

Lepelley, D.; Martin, M.: Condorcet's paradox for weak preference orderings, European Journal of Political Economy **17** (2001), pp. 163–177

Loomes, G.; Starmer, C.; Sugden; R.: Preference reversal: Information-processing effect or rational non-transitive choice?, Economic Journal **99** (1989), pp. 140–151.

Loomes, G.; Starmer, C.; Sugden, R.: Observing Violations of transitivity by experimental methods, Econometrica **59** (1991), pp. 425–439.

Matousek, J.: Lectures on discrete geometry, Springer Verlag (2002).

Nielsen, K.; Rigotti, L.: Revealed incomplete preferences, Papers 2205.08584, arXiv.org (2022).

Plott, C.R.: A notion of equilibrium and its possibility under the majority rule, American Economic Review **57** (1967), pp. 787–806.

Starmer, C.: Cycling with Rules of Thumb: An experimental test for a new form of non-transitive behaviour, Theory and Decision **46** (1999), pp. 139–157.

Tullock, G.: Why so much stability? Public Choice **37** (1981), pp. 189–204.

Tversky, A.: Intransitivity of preferences, Psychological Review **76** (1969), pp 31–48.

Tversky, A.; Shafir, E.: Choice under conflict: The dynamics of deferred decision. Psychological Science, **3** (1992), pp 358–361.

Van Deemen, A.: The probability of the paradox of voting for weak preference orderings, Social Choice and Welfare **16** (1999), pp. 171–182.

Appendix: political stability as d and m tend to infinity

The notion of majority equilibrium is generalized to ρ-majority equilibria for all $\rho \in (0,1)$. A ρ-majority equilibrium is an alternative for which there is no other alternative that is strictly preferred by more than $\rho \times 100\%$ of the voters.

Definition 7 *A* ***ρ-majority equilibrium*** *is an alternative $\bar{x} \in A$ such that for all alternatives $x \in A$,*

$$|\{\, i \in M \mid x \in P_i(\bar{x}) \,\}| \;\leq\; \rho m.$$

Let $X_{d,m} = \{x_{d,1}, \dots, x_{d,m}\}$ be a sample of m independent and identically distributed random variables in $\mathbb{S}^{d-1}$ for every for every d and m. We consider Euclidean preferences and study the relationship between the number of policy issues d and voters m on the one side and the degree of orderedness ensuring existence of ρ-majority equilibria as both d and m tend to infinity.

The average ideal point converges to the centre of the unit-sphere 0 for m tending to infinity independently of whether d tends to infinity or not. Indeed, the expected value of the average ideal point is zero $\mathbb{E}[(1/m)\sum_j x_{d,j}] = 0$ and the variance of the average ideal point converges to zero as $\mathbb{V}[(1/m)\sum_j x_{d,j}] = 1/m$. Therefore, there is no need to separate between the average ideal point and the centre as in Theorem 3.

Theorem 5 *For all ρ and $(X_{m,d})_{m,d\in\mathbb{N}}$ with probability one for d and m tending to infinity the centre of the unit-sphere* 0 *is a ρ-majority equilibrium provided $\alpha < 90°$.*

Proof: The idea of the proof can be explained in a few steps. The first step is to find a cone with vertex at zero and some angle such that: it contains a ρ-majority of voters; and, all cones with vertex at zero and smaller angles do not contain a ρ-majority of voters. However, we use the $(\rho m{+}1)\rho m/2$ smallest angles instead of $\rho m{+}1$ voters whose ideal points are contained in a cone with vertex at zero and the smallest angle. These $(\rho m{+}1)\rho m/2$ smallest angles need not be associated with $\rho m{+}1$ voters. Ideal points associated with these angles can be contained in a cone with vertex at zero and angle smaller than or equal to the smallest angle needed to contain a ρ-majority. The second step is to show that the angle needed for a cone with vertex at zero to contain ideal points associated with the $(\rho m{+}1)\rho m/2$ smallest angles converges to $\pi/2$ as d and m tend to infinity.

For two points x_i and x_j on $\mathbb{S}^{d-1}$ let $\angle_{ij} = \angle x_i 0 x_j \in [0,\pi]$ be the angle between x_i and x_j. Obviously, for m points, then there are $m(m{-}1)/2$ angles $(\angle_{ij})_{ij}$. According to Cai et al. (2013) for the error function erf : $[0,\pi]$ defined by

$$\operatorname{erf}(\angle) \;=\; \frac{2}{\sqrt{\pi}} \int_0^{\angle} e^{-t^2}\, dt,$$

the density $p(\angle)$ and the cumulative distribution $P(\angle)$ of angles are

$$\begin{cases} p(\angle) \;=\; \dfrac{1}{\sqrt{2\pi}} e^{-(\sqrt{d-2}(\pi/2-\angle))^2/2} \\[2ex] P(\angle) \;=\; 1-\dfrac{1}{2}\left[1{+}\operatorname{erf}\left(\dfrac{d-2}{2}\left(\dfrac{\pi}{2}-\angle\right)\right)\right]. \end{cases}$$

Therefore, the distribution of angles $\angle$ has mean $\pi/2$ and variance $1/(d{-}2)$, which converges to zero as d tends to infinity. According to Theorem 4 in Cai et al. (2013) the distribution of the normalized angles $(\sqrt{d{-}2}(\pi/2{-}\angle_{ij}))_{i<j}$ converges weakly to the standard normal distribution $N(0,1)$ as d and m tend to infinity. The half normal distribution with $\angle \in [0,\pi/2]$ has mean $\pi/2{+}(1/(d-2))\sqrt{\pi/2}$, which converges to $\pi/2$ as d tends to infinity, and variance $(1{-}2/\pi)/(d{-}2)$, which converges to zero as d tends to infinity.

For $\rho \in (0,1)$ consider the $(\lfloor \rho m \rfloor{+}1)\lfloor \rho m \rfloor/2$ smallest angles. Let $\lambda_m^\rho \in [0,1]$ be the ratio between the number of the $(\lfloor \rho m \rfloor{+}1)\lfloor \rho m \rfloor/2$ smallest angles and the number of angles $m(m{-}1)/2$,

$$\lambda_m^\rho \;=\; \frac{(\lfloor \rho m \rfloor{+}1)\lfloor \rho m \rfloor}{m(m{-}1)}.$$

Then λ_m^ρ converges to ρ^2 as m tends to infinity. Let $\angle_{d,m}^\rho \in [0,\pi]$ be defined by $P(\angle_{d,m}^\rho) = \lambda_m^\rho$. Clearly, $\rho^2 \gtreqless 1/2$ if and only if $\angle_{d,m}^\rho \gtreqless \pi/2$. Moreover, since P is continuous, if d tends to infinity, then $\angle_{d,m}^\rho$ converges to $\pi/2$ for all ρ because P is independent of m.

The average angle $\mathbb{E}_{d,m}^\rho[\angle_{d,m}^\rho]$ in the set of the $(\lfloor \rho m \rfloor{+}1)\lfloor \rho m \rfloor/2$ smallest angles converges to

$$\mathbb{E}_{d,m}^\rho[\angle_{d,m}^\rho] \;\approx\; \frac{1}{\lambda_m^\rho}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\angle_{d,m}^\rho} \angle\, e^{-(\sqrt{d}(\pi/2-\angle))^2/2}\, d\angle.$$

as d and m tend to infinity. Two cases are considered: $\lambda_m^\rho \leq 1/2$ so $\angle_{d,m}^\rho \geq \pi/2$; and, $\lambda_m^\rho \geq 1/2$ so $\angle_{d,m}^\rho \leq \pi/2$. In the first case it is used that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \angle e^{-(\sqrt{d}(\pi/2-\angle))^2/2}\, d[\angle_{d,m}^\rho,\pi] \\
&\qquad = \frac{\lambda_m^\rho}{2}\mathbb{E}_{d,m}^\rho(-\infty,\angle_{d,m}^\rho] + \frac{1-\lambda_m^\rho}{2}\mathbb{E}_{d,m}^\rho[\angle_{d,m}^\rho,\pi/2] + \frac{1}{2}\mathbb{E}_{d,m}^\rho[\pi/2,\infty) \\
&\qquad = \frac{\pi}{2}.
\end{aligned}$$

Since $\mathbb{E}_{d,m}^\rho(-\infty,\pi/2]$ and $\mathbb{E}_{d,m}^\rho[\pi/2,\infty)$ are means for half normal distributions,

$$\begin{cases}
\lim\limits_{d,m\to\infty} \mathbb{E}_{d,m}^\rho[\pi/2,\infty) &= \dfrac{\pi}{2} \\
\lim\limits_{d,m\to\infty} \mathbb{E}_{d,m}^\rho(-\infty,\angle_{d,m}^\rho] &= \dfrac{\pi}{2} \\
\lim\limits_{d,m\to\infty} \lambda_m^\rho &= \rho^2.
\end{cases}$$

Therefore, $\mathbb{E}_{d,m}^\rho(-\infty,\angle_{d,m}^\rho]$ converges to $\pi/2$ as d and m tend to infinity. In the second case it is used that

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \angle e^{-(\sqrt{d}(\pi/2-\angle))^2/2}\, d[\angle_{d,m}^\rho,\pi] \\
&\qquad = \frac{1}{2}\mathbb{E}_{d,m}^\rho(-\infty,\pi/2] + \frac{\lambda_m^\rho-1/2}{2}\mathbb{E}_{d,m}^\rho[\pi/2,\angle_{d,m}^\rho] + \frac{1-\lambda_m^\rho}{2}\mathbb{E}_{d,m}^\rho[\angle_{d,m}^\rho,\infty) \\
&\qquad = \frac{\pi}{2}.
\end{aligned}$$

Since $\mathbb{E}_{d,m}^\rho(-\infty,\pi/2]$ and $\mathbb{E}_{d,m}^\rho[\pi/2,\infty)$ are means for half normal distributions,

$$\begin{cases}
\lim\limits_{d,m\to\infty} \mathbb{E}_{d,m}^\rho(-\infty,\angle_{d,m}^\rho] &= \dfrac{\pi}{2} \\
\lim\limits_{d,m\to\infty} \mathbb{E}_{d,m}^\rho[\pi/2,(2\lambda_m^\rho-1)/2] &= \dfrac{\pi}{2} \\
\lim\limits_{d,m\to\infty} \lambda_m^\rho &= \rho^2.
\end{cases}$$

Therefore, $\mathbb{E}_{d,m}^\rho(-\infty,\angle_{d,m}^\rho]$ converges to $\pi/2$ as d and m tend to infinity.

For every $n \in N$ let $A_{d,m}^{\rho,n} \subset \mathbb{R}^d$ be a random sample of $n(n-1)/2$ angles in the set of the smallest $(\lfloor \rho m \rfloor + 1)\lfloor \rho m \rfloor/2$ angles. Then with probability one every angle in $A_{d,m}^{\rho,n}$ converges to $\pi/2$ as d and m tend to infinity. Since it is true for every $n \in \mathbb{N}$, it follows from the proof of Theorem 3 that with probability one for all $\alpha < \pi/2$ there is a ρ-majority equilibrium as d and m tend to infinity. □